数理解析研究所講究録
第 1611 巻 2008 年 122-127



# 均衡問題と解の近似について
# An equilibrium problem and approximation of its solutions

東京工業大学・大学院情報理工学研究科
木村泰紀 (Yasunori Kimura)
Department of Mathematical and Computing Sciences
Tokyo Institute of Technology

## 1 Introduction

Let $X$ be a set and $f : X \times X \to \mathbb{R}$. The equilibrium problem for $f$ is to find a point $x \in X$ such that

$$f(x, y) \geq 0 \quad \text{for all } y \in X,$$

and the set of its solutions is denoted by $EP(f)$. It is known that the equilibrium problem includes many kinds of important problems in various fields of applied mathematics such as minimization problems, saddle point problems, Nash equilibria in noncooperative games, fixed point problems, and others; see [4].

The existence of the solution for equilibrium problem has been discussed; for instance, see Fan [7], Takahashi [16], Blum and Oettli [4], Iusem and Sosa [11], and others. On the other hand, various types of approximating the solution has been proposed; see Flåm and Antipin [8], Combettes and Hirstaga [6], Iiduka and Takahashi [10], Tada and Takahashi [15], and others.

In this paper, we deal with a sequence of functions as an approximate of the function appearing in the original equilibrium problem. We assume convergence of a sequence of corresponding sets of solutions of equilibrium problems in the sense of Mosco. We obtain weak and strong convergence of a sequence of resolvents to a generalized projection onto the original set of solutions under certain conditions. Our main results are a generalized version of the results discussed in [12].

## 2 Preliminaries

Throughout this paper, we always deal with a real Banach space and denote it by $E$. We denote its norm by $\|\cdot\|$, its dual space by $E^*$, and for $x^* \in E^*$, the value of $x^*$

at $x \in E$ by $\langle x, x^* \rangle$. The norm of $E^*$ is also denoted by $\|\cdot\|$.

A Banach space $E$ is said to be strictly convex if $\|x + y\|/2 < 1$ for every $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. The norm of $E$ is said to be Gâteaux differentiable if for each $x, y \in E$ satisfying that $\|x\| = \|y\| = 1$, it holds that $(\|x + ty\| - \|x\|)/t$ converges as $t \to 0$, and in this case $E$ is said to be smooth. $E$ is said to have the Kadec-Klee property if a weakly convergent sequence $\{x_n\}$ of $E$ with a limit $x_0$ converges strongly to $x_0$ whenever $\{\|x_n\|\}$ converges to $\|x_0\|$. For more details, see [9, 17].

The normalized duality mapping $J : E \rightrightarrows E^*$ is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for $x \in E$. We know that $J$ is single-valued if $E$ is smooth. In this case, $J : E \to E^*$ is norm-to-weak* continuous. Moreover, if $E$ is reflexive and strictly convex, then $J$ is a bijection from $E$ onto $E^*$.

Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of a reflexive Banach space $E$. We define two subsets $\text{s-Li}_n C_n$ and $\text{w-Ls}_n C_n$ as follows: $x \in \text{s-Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to $x$ and that $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in \text{w-Ls}_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to $y$ and that $y_i \in C_{n_i}$ for all $n \in \mathbb{N}$. We define the Mosco convergence [13] of $\{C_n\}$ as follows: If $C_0$ satisfies that

$$C_0 = \operatorname*{s-Li}_n C_n = \operatorname*{w-Ls}_n C_n,$$

it is said that $\{C_n\}$ converges to $C_0$ in the sense of Mosco and we write $C_0 = \text{M-lim}_{n\to\infty} C_n$. For more details, see [3].

Let $C$ be a nonempty closed convex subset of a smooth, reflexive and strictly convex Banach space $E$. We consider a function $\phi : E \times E \to \mathbb{R}$ defined as

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for $x, y \in E$. It is easy to show that $\phi(x, y) \geq 0$ for all $x, y \in E$. From strict convexity of $E$, the function $\phi(\cdot, y)$ is a strictly convex function for every $y \in E$. Therefore, for arbitrarily fixed $y \in E$, a function $\phi(\cdot, y)|_C$ has a unique minimizer, say $x_y \in C$. Using this point, we define the generalized projection $\Pi_C$ such that $\Pi_C y = x_y$ for all $y \in E$. Notice that if $E$ is a Hilbert space, $\Pi_C$ coincides with the metric projection onto $C$ since $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$ in this case. For more details, see, for example, [1, 5, 14].

## 3 Convergence of resolvents for a sequence of functions

Let $E$ be a real Banach space and $C$ a nonempty convex subset of $E$. We assume that a function $f : C \times C \to \mathbb{R}$ satisfies the following conditions:

(E1) $f(x,x) = 0$ for every $x \in C$;
(E2) $f(x,y) + f(y,x) \leq 0$ for every $x, y \in C$;
(E3) $f(x,\cdot)$ is convex and lower semicontinuous for every $x \in C$.

In [12], we assume the following condition which is called upper hemicontinuity of $f$ with respect to the first variable;

(E4) $\limsup_{t \downarrow 0} f(ty + (1-t)x, y) \leq f(x,y)$ for every $x, y \in C$.

We shall assume the maximal monotonicity [4, 2] of $f$ instead of (E4) as follows:

(E5) for each $x \in C$ and $x^* \in E^*$, if $\langle z - x, x^* \rangle - f(z,x) \geq 0$ for all $z \in C$, then $\langle y - x, x^* \rangle + f(x,y) \geq 0$ for all $y \in C$.

**Theorem 1 (Aoyama-Kimura-Takahashi [2]).** *Let $E$ be a reflexive, smooth, and strictly convex Banach space and $C$ a nonempty closed convex subset of $E$. Let $f : C \times C \to \mathbb{R}$ satisfy the conditions* (E1), (E2), (E3), *and* (E5). *Then for every $x \in E$, there exists a unique $u \in C$ such that*

$$0 \leq f(u,y) + \langle y - u, Ju - Jx \rangle$$

*for all $y \in C$.*

This theorem guarantees that a resolvent $T_{rf}$ for $f : C \times C \to \mathbb{R}$ and $r > 0$ defined by

$$T_{rf} \colon E \ni x \mapsto \{u \in C : 0 \leq rf(u,y) + \langle y - u, Ju - Jx \rangle \, , \ \forall y \in C\} \subset C$$

is well defined as a single-valued mapping of $E$ into $C$. Namely, for every $x \in E$, $T_{rf}x$ is a unique point of $C$ which satisfies that

$$0 \leq rf(T_{rf}x, y) + \langle y - T_{rf}x, JT_{rf}x - Jx \rangle$$

for all $y \in C$.

On the other hand, it is easy to see that if $f$ satisfies the conditions (E1), (E2), (E3), and (E4), then $f$ also satisfies the condition (E5). Indeed, let $x \in C$, $x^* \in E^*$, and suppose that for $f$ satisfying (E5), $\langle z - x, x^* \rangle - f(z,x) \geq 0$ for all $z \in C$. Then, for arbitrarily chosen $y \in C$ and $0 < t < 1$, it follows that

$$\begin{aligned}
0 &= f(tx + (1-t)y, tx + (1-t)y) \\
&\leq tf(tx + (1-t)y, x) + (1-t)f(tx + (1-t)y, y) \\
&\leq t \langle tx + (1-t)y - x, x^* \rangle + (1-t)f(tx + (1-t)y, y) \\
&= t(1-t) \langle y - x, x^* \rangle + (1-t)f(tx + (1-t)y, y),
\end{aligned}$$

and thus $t \langle y - x, x^* \rangle + f(tx + (1-t)y, y) \geq 0$. Tending $t \to 1$, we have that

$$\langle y - x, x^* \rangle + f(x,y) \geq 0$$

by using (E4). Hence $f$ satisfies (E5).

Therefore, we obtain the following results, which generalize the results shown by the author in [12]. The proofs are the same as in [12].

**Theorem 2.** *Let $E$ be a reflexive, smooth, and strictly convex Banach space and $C$ a nonempty closed convex subset of $E$. Let $\{r_n\}$ be a positive real sequence such that $\lim_{n\to\infty} r_n = \infty$. Let $\{f_n\}$ be a sequence of functions of $C \times C$ into $\mathbb{R}$ satisfying the conditions* (E1), (E2), (E3), *and* (E5). *Let $C_0$ be a nonempty closed convex subset of $C$ satisfying the following conditions:*

(i) $C_0 \subset \text{s-Li}_n\, EP(f_n)$;
(ii) $\text{w-Ls}_n\, EP(f_n + g_{u_n^*}) \subset C_0$ *for every* $\{u_n^*\} \subset E^*$ *converging strongly to* $0$,

*where $g_{u^*} : C \times C \to \mathbb{R}$ is defined by $g_{u^*}(x, y) = \langle y - x, u^* \rangle$ for $x, y \in C$. Then, a sequence of resolvents $\{T_{r_n f_n} x\}$ converges weakly to $\Pi_{C_0} x \in C_0$ for every $x \in C$.*

**Theorem 3.** *Let $E$ be a reflexive, smooth, and strictly convex Banach space having the Kadec-Klee property. Let $C$, $\{r_n\}$, $\{f_n\}$ be the same as Theorem 2. Then, a sequence of resolvents $\{T_{r_n f_n} x\}$ converges strongly to $\Pi_{C_0} x \in C_0$ for every $x \in C$.*

Letting $f_n = f$ for all $n \in \mathbb{N}$, we deduce the following corollary.

**Corollary 1.** *Let $E$ be a reflexive, smooth, and strictly convex Banach space and $C$ a nonempty closed convex subset of $E$. Let $\{r_n\}$ be a positive real sequence such that $\lim_{n\to\infty} r_n = \infty$. Let $f$ be a function of $C \times C$ into $\mathbb{R}$ satisfying the conditions* (E1), (E2), (E3), *and* (E5). *Then, a sequence of resolvents $\{T_{r_n f} x\}$ converges weakly to $\Pi_{EP(f)} x \in EP(f)$ for every $x \in C$. Moreover, if $E$ has the Kadec-Klee property, then $\{T_{r_n f} x\}$ converges strongly to $\Pi_{EP(f)} x \in EP(f)$ for every $x \in C$.*

*Proof.* Let $f_n = f$ for all $n \in \mathbb{N}$ and $C_0 = EP(f)$. Then, it is obvious that the condition (i) in Theorem 2 is satisfied. For (ii), Let $\{u_n^*\}$ be a sequence of $E^*$ converging strongly to 0 and $v \in \text{w-Ls}_n\, EP(f + g_{u_n^*})$. Then, there exist a subsequence $\{n_i\}$ of $\mathbb{N}$ and a sequence $\{v_i\} \subset E$ such that $v_i \in EP(f + g_{u_{n_i}^*})$ and that $\{v_i\}$ converges weakly to $v$. Then, we have that

$$f(v_i, z) + g_{u_{n_i}^*}(v_i, z) = f(v_i, z) + \langle z - v_i, u_{n_i}^* \rangle \geq 0$$

for all $z \in C$. By (E2), it follows that $\langle z - v_i, u_{n_i}^* \rangle - f(z, v_i) \geq 0$ for $z \in C$ and using (E5), we obtain that

$$\langle y - v_i, u_{n_i}^* \rangle + f(v_i, y) \geq 0$$

for all $y \in C$. As $i \to \infty$, we have that

$$f(v, y) = \langle y - v, 0 \rangle + f(v, y) \geq 0$$

for all $y \in C$ and hence $v \in EP(f)$. Therefore $\text{w-Ls}_n\, EP(f + g_{u_n^*}) \subset EP(f) = C_0$ and (ii) holds. Using Theorems 2 and 3, we obtain the desired result. □

## References


[1] Y. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, Theory and applications of nonlinear operators of accretive and monotone type, Dekker, New York, 1996, pp. 15–50.

[2] K. Aoyama, Y. Kimura, and W. Takahashi, *Maximal monotone operators and maximal monotone functions for equilibrium problems*, J. Convex Anal., to appear.

[3] G. Beer, *Topologies on closed and closed convex sets*, Kluwer Academic Publishers Group, Dordrecht, 1993.

[4] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994), 123–145.

[5] D. Butnariu and A. N. Iusem, *Totally convex functions for fixed points computation and infinite dimensional optimization*, Applied Optimization, vol. 40, Kluwer Academic Publishers, Dordrecht, 2000.

[6] P. L. Combettes and S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal. **6** (2005), 117–136.

[7] K. Fan, *Some properties of convex sets related to fixed point theorems*, Math. Ann. **266** (1984), 519–537.

[8] S. D. Flåm and A. S. Antipin, *Equilibrium programming using proximal-like algorithms*, Math. Programming **78** (1997), 29–41.

[9] K. Goebel and S. Reich, *Uniform convexity, hyperbolic geometry, and nonexpansive mappings*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 83, Marcel Dekker Inc., New York, 1984.

[10] H. Iiduka and W. Takahashi, *Relations between equations of set-valued operators and equilibrium problems*, Proceedings of the Fourth International Conference on Nonlinear Analysis and Convex Analysis (Tokyo, Japan) (W. Takahashi and T. Tanaka, eds.), Yokohama Publishers, 2007, pp. 163–172.

[11] A. N. Iusem and W. Sosa, *New existence results for equilibrium problems*, Nonlinear Anal. **52** (2003), 621–635.

[12] Y. Kimura, *Equilibrium problems and convergence of resolvents for a sequence of functions*, Proceedings of the International Symposium on Banach and Function Spaces (Kitakyushu, Japan) (M. Kato and L. Maligranda, eds.), Yokohama Publishers, to appear.

[13] U. Mosco, *Convergence of convex sets and of solutions of variational inequalities*, Adv. in Math. **3** (1969), 510–585.

[14] S. Reich, *Constructive techniques for accretive and monotone operators*, Applied nonlinear analysis (Proc. Third Internat. Conf., Univ. Texas, Arlington, Tex., 1978), Academic Press, New York, 1979, pp. 335–345.

[15] A. Tada and W. Takahashi, *Strong convergence theorem for an equilibrium problem and a nonexpansive mapping*, Proceedings of the Fourth International Con-

ference on Nonlinear Analysis and Convex Analysis (Tokyo, Japan) (W. Takahashi and T. Tanaka, eds.), Yokohama Publishers, 2007, pp. 609–617.

[16] W. Takahashi, *Fixed point, minimax, and Hahn-Banach theorems*, Nonlinear functional analysis and its applications, Part 2 (Berkeley, Calif., 1983), Proc. Sympos. Pure Math., vol. 45, Amer. Math. Soc., Providence, RI, 1986, pp. 419–427.

[17] ———, *Nonlinear functional analysis: fixed point theory and its applications*, Yokohama Publishers, Yokohama, 2000.